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Lower semicontinuity of the solution map to a parametric vector variational inequality

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Abstract This paper is concerned with the study of solution stability of a parametric vector variational inequality, where mappings may not be strongly monotone. Under some requirements that the operator of a unperturbed problem is monotone or it satisfies degree conditions then we show that the solution map of a parametric vector variational inequality is lower semicontinuous.

Keywords Parametric vector variational inequality · Strict monotonicity · Degree theory · Lower semicontinuity

1 Introduction

Vector variational inequality (VVI, for short) was introduced by Giannessi [6] in 1980. Later on, VVI and its many extensions was studied by Chen [2,3], Kien [11], Lee [13] and Yang [16] (see also the references given therein). The main topic of these papers is to establish existence theorems.

Nowadays, VVIs appear in many important problems from theory to applications such as vector optimization theory, economics and transportation. In such applications it is important to understand behaviors of a solution of a vector variational inequality when the problem's data vary. In other words we wish to know properties of solutions of the so-called parametric vector variational inequalities when parameters vary. One of our interests is to investigate the continuity of the solution map of such a problem.

Let us assume that R^n and R^m are Euclidian spaces with the scalar product $\langle , \rangle$ and the Euclidean norm $\| \cdot \|$. We shall use the notation

$$R^m_+ = \{x = (x_1, x_2, \ldots, x_m) : x_i \geq 0 \text{ for all } i\}.$$

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Let $L(R^n, R^m)$ be the set of continuous mappings from R^n into R^m. Suppose M, Λ are metric spaces, $K : \Lambda \to 2^{R^n}$ is a set-valued mapping with closed convex values and $f_i : M \times R^n \to R^n (i = 1, 2, \ldots, m)$ are continuous maps. Let $f : M \times R^n \to L(R^n, R^m)$ be a mapping which is defined by

$$f(\mu, x)(u) = (\langle f_1(\mu, x), u\rangle, \langle f_2(\mu, x), u\rangle, \ldots, \langle f_m(\mu, x), u\rangle).$$

Let C be a closed convex cone in R^m such that $\mathrm{Int}C \neq \emptyset$, where $\mathrm{Int}C$ denotes the interior of C.

The parametric vector variational inequality involving the set $K(\lambda)$, the mapping $f(\mu, \cdot)$ and the cone C is the problem of finding $x = x(\mu, \lambda) \in K(\lambda)$ satisfying the condition

$$f(\mu, x)(y - x) \notin -\mathrm{Int}C \quad \text{for all} \quad y \in K(\lambda). \tag{1}$$

We shall denote problem Eq. (1) briefly by $\mathrm{VVI}(f(\mu, \cdot), K(\lambda))$ and $S(\mu, \lambda)$ stands for its solution set corresponding to parameter (μ, λ). Thus $S : M \times \Lambda \to 2^{R^n}$ is a set-valued map which is called the solution map of Eq. (1). Given a parameter $(\mu_0, \lambda_0) \in M \times \Lambda$, we assume that $S(\mu_0, \lambda_0) \neq \emptyset$, that is, there is a point $x_0 \in S(\mu_0, \lambda_0)$ such that

$$f(\mu_0, x_0)(y - x_0) \notin -\mathrm{Int}C \quad \text{for all} \quad y \in K(\lambda_0). \tag{2}$$

Our main concern is to investigate the behaviour of $S(\mu, \lambda)$ when (μ, λ) vary around (μ_0, λ_0).

Recently Lee et al. [14] have shown that if f is strongly monotone and the set-valued map K has the Aubin property at every point (λ_0, x_0), where $x_0 \in S(\mu_0, \lambda_0)$, then the solution map $S(\mu, \lambda)$ is Hölder-Lipschitz continuous with respect to (μ, λ) (see [14, Theorem 5.1]). To obtain this result the authors had to use the Banach contractive theorem and the results given by [17].

The situation will become complicated when the hypothesis on the strong monotonicity is dropped. In this case, it seems difficulty to use the techniques in the proof of Theorem 5.1 in [14], because the strong monotonicity of f played an essential role in establishing the proof.

The aim of this paper is to establish some results on the solution stability of the parametric vector variational inequality Eq. (1) without strong monotonicity of f. In order to obtain the result we have to derive a new scheme for the proof by using some facts on the degree theory and a result on a relation between the Aubin property and the Hölder property of multifunctions K. Using this scheme, we show that if the mapping $f(\mu_0, \cdot)$ of the unpurturbed problem is strictly monotone or it satisfies some requirement related to degree theory, then the solution map is lower semicontinous.

The rest of the paper consists of two sections. Section 1 is devoted to a result on the lower semicontinuity of the solution set under hypotheses relating to strict monotonicity of mappings of the unperturbed problem. In Sect. 2 we give sufficient conditions for the lower semicontinuity of the solution map with requirements on the degree of mappings.

2 A monotone-operator approach

In this section we first establish some auxiliary results on a relation between the solution set of a vector variational inequality and the solution set of a scalar variational inequality.

Let us assume that Ω is a closed convex subset in R^n and C be a closed convex cone in R^m with $\mathrm{Int}C \neq \emptyset$. We shall denote by $L(R^n, R^m)$ the set of all continuous mapping from R^n into R^m and C^* the polar cone of C, that is

$$C^* = \{z^* \in R^m : \langle z^*, z\rangle \geq 0 \quad \text{for all} \quad z \in C\}.$$

Putting $C^*_+ = C^* \setminus \{0\}$ we obtain from the bipolar theorem (see, e.g., [8]) that

$$z \in -\mathrm{Int}C \Longleftrightarrow \langle z^*, z\rangle < 0 \quad \text{for all} \quad z^* \in C^*_+. \tag{3}$$

Suppose $h_i : \Omega \to R^n (i = 1, 2 \ldots, m)$ are continuous mappings and $h : \Omega \to L(R^n, R^m)$ is defined by

$$h(x)(u) = (\langle h_1(x), u\rangle, \langle h_2(x), u\rangle, \ldots, \langle h_m(x), u\rangle).$$

Consider the following VVI(h, Ω):

$$\text{Find } x_0 \in \Omega \text{ such that } h(x_0)(x - x_0) \notin -\mathrm{Int}C \quad \text{for all} \quad x \in \Omega. \tag{4}$$

In other form, Eq. (4) is equivalent to the problem

$$\text{Find } x_0 \in \Omega \text{ such that } \{h(x_0)(x - x_0) : x \in \Omega\} \cap (-\mathrm{Int}C) = \emptyset. \tag{5}$$

For each $\xi \in C^* \setminus \{0\}$ we consider the following variational inequality VI$(\xi h, \Omega)$:

$$\text{Find } x_0 \in \Omega \text{ such that } \sum_{i=1}^{m} \xi_i \langle h_i(x_0), x - x_0\rangle \geq 0 \quad \text{for all} \quad x \in \Omega. \tag{6}$$

This problem can be formulated by the term of generalized equations

$$\text{Find } x_0 \in \Omega \text{ such that } 0 \in \sum_{i=1}^{m} \xi_i h_i(x_0) + N_\Omega(x_0), \tag{7}$$

where $N_\Omega(x_0)$ is the normal cone of the set Ω at x_0 which defined by

$$N_\Omega(x) = \begin{cases} \{x^* \in R^n : \langle x^*, y - x\rangle \leq 0 \ \forall y \in \Omega\} & \text{if } x \in \Omega \\ \emptyset, \quad \text{otherwise.} & \end{cases}$$

Let us denote by S(VVI) and S(VI)$_\xi$ the solution sets of Eqs. (4) and (6), respectively. The following lemma gives a relation between the solution sets of Eqs. (4) and (6).

Lemma 2.1 (C. f. [13, Theorem 2.1]) *Suppose h_i are continuous for $i = 1, 2, \ldots, m$. The following assertions hold:*

(a)

$$\bigcup_{\xi \in C^*_+} \mathrm{S(VI)}_\xi = \mathrm{S(VVI)}. \tag{8}$$

(b) S(VVI) *is a closed set.*

Proof (a) Suppose x_0 is a point of $\bigcup_{\xi \in C^*_+} \mathrm{S(VI)}_\xi$. Then there exists $\xi \in C^*_+$ such that x_0 is a solution of VI$(\xi h, \Omega)$, that is Eq. (6) holds. By Eq. (3), it follows that Eq. (4) is satisfied. As x_0 is arbitrary, we have $\bigcup_{\xi \in C^*_+} \mathrm{S(VI)}_\xi \subseteq \mathrm{S(VVI)}$. Conversely, take any $x_0 \in \mathrm{S(VVI)}$. By Eq. (5) we have

$$\{h(x_0)(x - x_0) : x \in \Omega\} \cap (-\mathrm{Int}C) = \emptyset.$$

By the separation theorem (see [7, Theorem 1, p. 163]) there exists a functional $\xi \in R^m \setminus \{0\}$ such that

$$\langle \xi, h(x_0)(x - x_0)\rangle \geq \langle \xi, u\rangle$$

for all $x \in \Omega$ and $u \in -\mathrm{Int}C$. Put $x = x_0$ we have $\langle \xi, u \rangle \leq 0$ for all $u \in -\mathrm{Int}C$. This implies that $\langle \xi, u \rangle \geq 0$ for all $u \in C$. Consequently, $\xi \in C^*_+$. Thus we have shown that there exists $\xi \in C^*_+$ such that

$$\langle \xi, h(x_0)(x - x_0) \rangle \geq 0 \quad \text{for all} \quad x \in \Omega.$$

Hence $x_0 \in \bigcup_{\xi \in C^*_+} \mathrm{S(VI)}_\xi$. It follows that $\bigcup_{\xi \in C^*_+} \mathrm{S(VI)}_\xi \supseteq \mathrm{S(VVI)}$ and Eq. (8) is obtained.

(b) Since $A := R^n \setminus (-\mathrm{Int}C)$ is a closed set, the set

$$D(x) := \{y \in \Omega : f(y)(x - y) \in A\}$$

is closed. It is clear that $\mathrm{S(VVI)} = \cap_{x \in \Omega} D(x)$. Hence S(VVI) is a closed set. □

We now return to problem Eq. (1). For each $\xi \in C^*_+$ we denote by $S_\xi(\mu, \lambda)$ the solution set of the generalized equation

$$0 \in \sum_{i=1}^{m} \xi_i f_i(\mu, x) + N_{K(\lambda)}(x).$$

By lemma 2.1, for each $(\mu, \lambda) \in M \times \Lambda$, we have $\bigcup_{\xi \in C^*_+} S_\xi(\mu, \lambda) = S(\mu, \lambda)$.

Given a parameter $(\mu_0, \lambda_0) \in M \times \Lambda$, we assume that $S(\mu_0, \lambda_0) \neq \emptyset$. Taking any $x_0 \in S(\mu_0, \lambda_0)$, we see that $(\lambda_0, x_0) \in \mathrm{Grph}K$ and Eq. (2) hods. By Lemma 2.1, there exists a point $\xi^0 \in C^*_+$ such that

$$0 \in \sum_{i=1}^{m} \xi^0_i f_i(\mu_0, x_0) + N_{K(\lambda_0)}(x_0). \tag{9}$$

Recall that the set-valued mapping $K : \Lambda \to 2^{R^n}$ has the Aubin property of order $\alpha > 0$ at a point $(\lambda_0, x_0) \in \mathrm{grph}K$ if there exist positive constants k, ϵ_0 and β_0 such that

$$K(\lambda') \cap (x_0 + \epsilon_0 \bar{B}) \subseteq K(\lambda) + kd(\lambda', \lambda)^\alpha \bar{B} \quad \forall \lambda', \lambda \in B(\lambda_0, \beta_0). \tag{10}$$

Here $\bar{B}$ is the closed unit ball of R^n and $B(\lambda_0, \beta)$ is an open ball with center at λ_0 and radius β, in the metric space Λ. If $K(\cdot)$ satisfies property Eq. (10) for $\alpha = 1$ then $K(\cdot)$ is said to be pseudo-Lipschitz continuous at (λ_0, x_0). Let E be a subset in R^n and $g : E \to R^n$ be a mapping. We say that g strictly monotone on E if for any $x_1, x_2 \in E$ with $x_1 \neq x_2$, one has $\langle g(x_1) - g(x_2), x_1 - x_2 \rangle > 0$.

A multifunction $P : X \to 2^Y$ from a topological space X to a topological space Y is said to be lower semicontinuous at $x_0 \in X$ if for any open set V in Y satisfying $P(x_0) \cap V \neq \emptyset$, there exists a neighborhood U of x_0 such that $P(x) \cap V \neq \emptyset$ for all $x \in U$.

We are ready to state the first result.

Theorem 2.2 *Suppose C is a closed convex cone in R^m_+, $S(\mu_0, \lambda_0)$ is bounded, X_0 is a neighborhood of $S(\mu_0, \lambda_0)$ and $M_0 \times \Lambda_0$ is a neighborhood of (μ_0, λ_0). Let $f_i : M_0 \times X_0 \to R^n (i = 1, 2, \ldots, m)$ be continuous mappings and $K : \Lambda_0 \to 2^{R^n}$ be a multifunction which satisfy conditions:*

(i) $f_i(\mu_0, \cdot) : X_0 \cap K(\lambda_0) \to R^n$ is strictly monotone for all $i = 1, 2, \ldots, m$;
(ii) for each $x_0 \in S(\mu_0, \lambda_0)$, K has the Aubin property of order $\alpha > 0$ at the point (λ_0, x_0).

Then there exist a neighborhood $U_0 \times V_0$ of (μ_0, λ_0) and a bounded open neighborhood Q_0 of $S(\mu_0, \lambda_0)$ such that the following assertions hold:

(a) *The solution map $S : U_0 \times V_0 \to 2^{Q_0}$ has nonempty values.*
(b) *The solution map S is lower semicontinuous at (μ_0, λ_0).*

Proof Note that since $f_i(\mu_0, \cdot)$ is continuous, $S(\mu_0, \lambda_0)$ is closed. Hence $S(\mu_0, \lambda_0)$ is a compact set. Taking any $x_0 \in S(\mu_0, \lambda_0)$, we see that $(\lambda_0, x_0) \in \mathrm{Grph}K$ and there exists $\xi^0 \in C_+^*$ such that Eq. (9) is fulfilled. By (ii), there exist constants k, ϵ_0 and β_0 such that Eq. (10) holds. For each $\epsilon > 0$ we put

$$K_\epsilon(\lambda) = K(\lambda) \cap (x_0 + \epsilon \bar{B}). \tag{11}$$

The following lemma plays an important role in the proof of our theorem which is an extension of Lemma 2.3 in [1].

Lemma 2.3 *For any ϵ in $(0, \epsilon_0]$ and any β with $0 < \beta < \min\{\beta_0, (\frac{\epsilon}{4k})^{1/\alpha}\}$, the multifunction K_ϵ defined by Eq. (11), is Hölder continuous with constant $5k$ on the ball $x_0 + \beta \bar{B}$, that is*

$$K_\epsilon(\lambda') \subseteq K_\epsilon(\lambda) + 5kd(\lambda', \lambda)^\alpha \bar{B}. \tag{12}$$

for all $\lambda, \lambda' \in B(\lambda_0, \beta)$.

Proof We shall use similar arguments as in the proof of [1].

Put $\lambda' = \lambda_0$ in Eq. (10), we obtain that, for all $\lambda \in B(\lambda_0, \beta)$ there exists a point $x_\lambda \in K(\lambda)$ such that

$$\|x_\lambda - x_0\| \le kd(\lambda_0, \lambda)^\alpha < \epsilon/2. \tag{13}$$

This implies that $K_\epsilon(\lambda)$ is nonempty for all $\lambda \in B(\lambda_0, \beta)$. Take any $\lambda', \lambda \in B(\lambda_0, \beta)$ and $x' \in K_\epsilon(\lambda')$. We have to show that there exists a point $x'' \in K_\epsilon(\lambda)$ such that

$$\|x' - x''\| \le 5kd(\lambda', \lambda)^\alpha \tag{14}$$

which proves Eq. (12).

It follows from Eq. (10) that there exists $x \in K(\lambda)$ such that

$$\|x' - x\| \le kd(\lambda', \lambda)^\alpha \le 5kd(\lambda', \lambda)^\alpha.$$

If $x \in x_0 + \epsilon \bar{B}$ then $x \in K_\epsilon(\lambda)$. By putting $x'' = x$ we obtain Eq. (14).

Let us assume that $x \notin x_0 + \epsilon \bar{B}$. This means that

$$r := \|x - x_0\| > \epsilon.$$

Choose $x_\lambda \in K(\lambda)$ such that Eq. (13) holds. By the convexity of $K(\lambda)$, the segment $[x, x_\lambda] \subset K(\lambda)$. Take a point $x'' \in [x, x_\lambda]$ such that $\|x'' - x_0\| = \epsilon$, belongs to $K(\lambda)$. Hence $x'' \in K_\epsilon(\lambda)$. Note that such a point x'' always exists. Namely, $x'' = (1-t)x + tx_\lambda$, where $t \in (0, 1)$.

Put $\rho = \|x - x''\|$, $d = r - \epsilon$. Then we have

$$\epsilon = \|x'' - x_0\| = \|(1-t)(x - x_0) + t(x_\lambda - x_0)\| \le (1-t)r + t\|x_\lambda - x_0\|.$$

This implies that

$$t(r - \|x_\lambda - x_0\|) \le r - \epsilon.$$

Hence

$$t \le \frac{r - \epsilon}{r - \epsilon/2}.$$

From this relation,

$$\rho := \|x'' - x\| = t\|x - x_\lambda\| \le t(\|x - x_0\| + \|x_\lambda - x_0\|) \\ \le t(r + \epsilon/2) \le (r + \epsilon/2)\frac{r - \epsilon}{r - \epsilon/2}.$$

It follows that $\|x'' - x\| = \rho \le 4(r - \epsilon) = 4d$. Since $d \le \|x - x'\|$, $\|x'' - x\| \le 4\|x - x'\|$. Finally we have

$$\|x' - x''\| \le \|x'' - x\| + \|x - x'\| \le 5\|x - x'\| \le 5kd(\lambda', \lambda)^\alpha.$$

The lemma is proved.

We now Choose positive constants s and δ such that $x_0 + s\bar{B} \subset (x_0 + \epsilon_0\bar{B}) \cap X_0$, $kd(\lambda_0, \lambda)^\alpha < s$ for all $\lambda \in B(\lambda_0, \delta) \subset B(\lambda_0, \beta_0)$. Hence Eq. (10) implies

$$K(\lambda') \cap (x_0 + s\bar{B}) \subseteq K(\lambda) + kd(\lambda, \lambda')^\alpha \bar{B} \tag{15}$$

for all $\lambda, \lambda' \in B(\lambda_0, \delta)$.

Choose a number β such that $0 < \beta < \min\{\delta, (\frac{s}{4k})^{\frac{1}{\alpha}}\}$. By Lemma 2.3, we have

$$K(\lambda') \cap (x_0 + s\bar{B}) \subseteq K(\lambda) \cap (x_0 + s\bar{B}) + 5kd(\lambda', \lambda)^\alpha \bar{B} \tag{16}$$

for all $\lambda, \lambda \in B(\lambda_0, \beta)$. Thus for all $\lambda', \lambda \in B(\lambda_0, \beta)$, Eqs. (15) and (16) are fulfilled.

Putting $\lambda' = \lambda_0$ in Eq. (15) we see that for each $\lambda \in B(\lambda_0, \beta)$ there exists $z_\lambda \in K(\lambda)$ such that $\|z_\lambda - x_0\| \le kd(\lambda, \lambda_0)^\alpha < s$. Consequently $K(\lambda) \cap B(x_0, s) \ne \emptyset$ for all $\lambda \in B(\lambda_0, \beta)$.

For each $(\mu, \lambda) \in M_0 \times B(\lambda_0, \beta)$ we consider the generalized equation

$$0 \in g(\mu, x) + N_{K(\lambda)\cap\bar{B}(x_0,s)}(x)$$

where $g(\mu, x) := \sum_{i=1}^m \xi_i^0 f_i(\mu, x)$. We claim that there exists a neighborhood $U_0 \times V_0$ of (μ_0, λ_0) such that

$$0 \notin \big(g(\mu, \cdot) + N_{K(\lambda)\cap\bar{B}(x_0,s)}(\cdot)\big)(\partial B(x_0, s)) \tag{17}$$

for all $(\mu, \lambda) \in U_0 \times V_0$, where $\partial B(x_0, s)$ is the boundary of $B(x_0, s)$. Indeed, suppose the assertion is false. Then we can find sequences $\mu_n \to \mu_0$, $\lambda_n \to \lambda_0$ and $\{x_n\} \subset \partial B(x_0, s) \cap K(\lambda_n)$ such that

$$\langle g(\mu_n, x_n), z - x_n\rangle \ge 0 \quad \forall\, x_n \in K(\lambda_n) \cap \bar{B}(x_0, s). \tag{18}$$

Since $\partial B(x_0, s)$ is a compact set, we can assume that $x_n \to \bar{x}$. Substituting $\lambda' = \lambda_n$, $\lambda = \lambda_0$ into Eq. (16), we see that, for each n, there exists $y_n \in K(\lambda_0) \cap \bar{B}(x_0, s)$ such that

$$\|x_n - y_n\| \le 5kd(\lambda_n, \lambda_0)^\alpha.$$

Since $K(\lambda_0) \cap \bar{B}(x_0, s)$ is compact, without loss of generality we may assume that $y_n \to y_0 \in K(\lambda_0)\cap\bar{B}(x_0, s)$. From the above, we have $x_n \to y_0$. Hence $\bar{x} = y_0 \in K(\lambda_0)\cap\bar{B}(x_0, s)$. Putting $\lambda' = \lambda_0, \lambda = \lambda_n$ in Eq. (16), we see that for each n there exists a point $z_n \in K(\lambda_n) \cap \bar{B}(x_0, s)$ such that $z_n \to x_0$. Putting $z = z_n$ in Eq. (18) and letting $n \to \infty$ we obtain $\langle g(\mu_0, \bar{x}), x_0 - \bar{x}\rangle \ge 0$. Since $f_i(\mu_0, \cdot)$ is strictly monotone and $\xi^0 \in R_+^n \setminus \{0\}$, we see that $g(\mu_0, \cdot)$ is also strictly monotone. It is noted that $x_0 \ne \bar{x}$. Hence we have

$$\langle g(\mu_0, x_0), x_0 - \bar{x}\rangle > \langle g(\mu_0, \bar{x}), x_0 - \bar{x}\rangle \ge 0.$$

This contradicts the fact that x_0 satisfies Eq. (9). Thus the claim is proved.

We now can choose neighborhoods $U \subset M_0$ of μ_0 and $V \subset B(\lambda_0, \beta)$ of λ_0 such that Eq. (17) is valid. For each $(\mu, \lambda) \in U \times V$ we consider the generalized equation

$$0 \in g(\mu, x) + N_{K(\lambda)\cap\bar{B}(x_0,s)}(x). \tag{19}$$

Since $g(\mu, \cdot)$ is continuous and $K(\lambda) \cap \bar{B}(x_0, s)$ is compact, equation Eq. (19) has a solution $\hat{x} = \hat{x}(\mu, \lambda) \in K(\lambda) \cap \bar{B}(x_0, s)$. By Eq. (17), $\hat{x} \in \mathrm{int}\bar{B}(x_0, s)$. Hence

$$N_{K(\lambda)\cap\bar{B}(x_0,s)}(x) = N_{K(\lambda)}(\hat{x}).$$

Consequently, $\hat{x}$ is a solution of the equation

$$0 \in g(\mu, x) + N_{K(\lambda)}(x).$$

This implies that $\hat{x} \in S_{\xi^0}(\mu, \lambda) \cap B(x_0, s)$. Hence we also have $S(\mu, \lambda) \cap B(x_0, s) \neq \emptyset$.

So far we have shown that for each $x_0 \in S(\mu_0, \lambda_0)$ there exist a open ball $B(x_0, s)$ and a neighborhood $U_{x_0} \times V_{x_0}$ of (μ_0, λ_0) such that $S(\mu, \lambda) \cap B(x_0, s) \neq \emptyset$ for all $(\mu, \lambda) \in U_{x_0} \times V_{x_0}$. As $S(\mu_0, \lambda_0)$ is a compact set, there exists $x_1, x_2, \ldots, x_k$ such that $S(\mu_0, \lambda_0) \subseteq \cup_{i=1}^{n} B(x_i, s_i)$. Put $Q_0 = \cup_{i=1}^{k} B(x_i, s_i)$, $U_0 = \cap_{i=1}^{k} U_{x_i}$ and $V_0 = \cap_{i=1}^{k} V_{x_i}$. It is clear that U_0, V_0 and Q_0 satisfy assertion (a) of the theorem. It remains to prove assertion (b). Suppose W is a open set in Q_0 such that $S(\mu_0, \lambda_0) \cap W \neq \emptyset$. Note that $W = Q_0 \cap G$, where G is a open set in R^n. Take $x_0 \in S(\mu_0, \lambda_0) \cap W$. By (ii) there exist constants k, ϵ_0 and β_0 such that Eq. (10) is fulfilled. We can choose ϵ_0 such that $\bar{B}(x_0, \epsilon_0) \subset W$. We now use the same arguments as in the proof of part (a) to show that there exist a ball $B(x_0, \hat{s}) \subset B(x_0, \epsilon_0)$, a neighborhood $\hat{U} \times \hat{V} \subset U_0 \times V_0$ of (μ_0, λ_0) such that $S(\mu, \lambda) \cap B(x_0, \hat{s}) \neq \emptyset$ for all $(\mu, \lambda) \in \hat{U} \times \hat{V}$. This implies that $S(\mu, \lambda) \cap W \neq \emptyset$ for all $(\mu, \lambda) \in \hat{U} \times \hat{V}$. Consequently, S is lower semicontinuous at (μ_0, λ_0). The proof of the theorem is complete. □

In order to make readers to illustrate the Theorem 2.2, we give the following example.

Example 2.4 Let $(\mu_0, \lambda_0) = (-1, 1)$, $M_0 \times \Lambda_0 \subset R^2$ be a neighborhood of (μ_0, λ_0) and $X_0 = R^2$. Let $f = (f_1, f_2)$ and $f_1, f_2 : M_0 \times X_0 \to R^2$ be defined by

$$f_1(\mu, x) = (x_1, \mu x_2 + x_2^2),\ f_2(\mu, x) = (x_1 + (1-\mu)x_2, x_2), \quad x = (x_1, x_2)$$

and $K : \Lambda_0 \to R^2$ defined by

$$K(\lambda) = \{(x_1, x_2) : x_2 \geq -1, x_1 + x_2 = \lambda\}.$$

Then all conditions of Theorem 2.2 are satisfied and $u_0 = (0, 1)$ is a solution of VVI$(f(\mu_0, \cdot), K(\lambda_0))$;

In fact, we have $f_1(\mu_0, x) = (x_1, -x_2 + x_2^2)$ and

$$K(\lambda_0) = \{(x_1, x_2) : x_2 \geq 1, x_1 + x_2 = 1\}.$$

For any $u = (u_1, u_2)$, $v = (v_1, v_2) \in K(\lambda_0)$, we see that $u_2 + v_2 \geq 2$. Hence

$$\langle f_1(\mu_0, u) - f_1(\mu_0, v), u - v\rangle = (u_1 - v_1)^2 + (u_2 - v_2)^2(u_2 + u_2 - 1) > 0$$

whenever $u \neq v$. This implies that $f_1(\mu_0, \cdot)$ is strictly monotone. When $\mu_0 = -1$ we have $f_2(\mu_0, x) = (x_1, x_2)$ which is also strictly monotone. Hence condition (i) of Theorem 2.2 is fulfilled.

It is clear that $f_1(\cdot, \cdot)$ and $f_2(\cdot, \cdot)$ are continuous. On the other hand, for each $\lambda \in \Lambda_0$, $K(\lambda)$ is a closed convex set. By [15], the map $K(\cdot)$ is Lipshitz continuous. Hence condition (ii) in Theorem 2.2 is valid. Thus all conditions of Theorem 2.2 are fulfilled.

Taking $\xi = (1, 0)$ we see that if $u \in K(\lambda)$ is a solution of VI$(f_1(\mu, \cdot), K(\lambda))$ then it is also a solution of VVI$(f(\mu, x), K(\lambda))$.

On the other hand, $u \in K(\lambda)$ is a solution of VI$(f_1(\mu, \cdot), K(\lambda))$ iff

$$u = \Pi_{K(\lambda)}(u - \rho f_1(\mu, u))$$

for some $\rho > 0$. Here $\Pi_{K(\lambda)}(x)$ stands for the metric projection of a point $x \in R^2$ onto the set $K(\lambda)$. Let $x = (x_1, x_2)$ be any point in R^2. By a simple computation, we obtain

$$\Pi_{K(\lambda)}(x) = \left(\frac{\lambda + x_1 - x_2}{2}, \frac{\lambda + x_2 - x_1}{2}\right).$$

It follows that

$$\Pi_{K(\lambda)}(x - \rho f(\mu, x)) = \frac{1}{2}\big(\lambda + (1-\rho)x_1 - (1-\mu\rho)x_2 + \rho x_2^2, \lambda \\ -(1-\rho)x_1 + (1-\mu\rho)x_2 - \rho x_2^2\big).$$

Hence

$$(x_1, x_2) = \Pi_{K(\lambda)}(x - \rho f(\mu, x))$$

if and only if

$$\begin{cases} \lambda + (1-\rho)x_1 - (1-\mu\rho)x_2 + \rho x_2^2 = 2x_1 \\ \lambda - (1-\rho)x_1 + (1-\mu\rho)x_2 - \rho x_2^2 = 2x_2. \end{cases}$$

It is equivalent to

$$\begin{cases} x_2^2 + (1+\mu)x_2 - \lambda = 0 \\ x_1 + x_2 = \lambda. \end{cases} \tag{24}$$

When $(\mu_0, \lambda_0) = (-1, 1)$ the system has the unique solution $u_0 = (0, 1) \in K(\lambda_0)$.

□

3 A degree-theoretic approach

In this section we shall provide sufficient conditions for the lower semicontinuity of the solution map of problem Eq. (1). Here the conditions are related to the degree of mappings which guarantee the lower semicontinuity of the solution map S. Such a degree-theoretic approach for the scalar case has been used by [10].

Before stating our result we recall some notions and facts of the degree theory. The notions and events of the degree theory can be found in [4,5,12,18].

Let D be an open bounded set in R^n. We denote by ∂D the boundary of D and $\bar{D}$ the closure of D. Let $C^1(\bar{D}) = C^1(D) \cap C(\bar{D})$, where $C^1(D)$ is the set of all continuously differentiable functions $\phi : D \to R^n$ and $C(\bar{D})$ is the set of all continuous functions on $\bar{D}$.

For each $\phi \in C(\bar{D})$ we put $\|\phi\| = \max_{x\in\bar{D}} \|\phi(x)\|$.

We will denote by dist(x, A) the distance form a point $x \in R^n$ to a set $A \subset R^n$.

If $\phi \in C^1(\bar{D})$, $J_\phi(x) = \det(\text{grad}\phi(x))$ and $Z_\phi = \{x \in \bar{D} : J_\phi(x) = 0\}$ which is called the crease of ϕ.

It is well known that if $\phi \in C^1(\bar{D})$ and $p \notin \phi(Z_\phi)$ then the set $\phi^{-1}(p)$ is finite (see, for instance [12, Theorem 1.1.2]).

Definition 3.1 (a) Let $\phi \in C^1(\bar{D})$ and $p \notin \phi(Z_\phi) \cup \phi(\partial D)$. The degree of ϕ at p with respect to D is defined by

$$\deg(\phi, D, p) := \sum_{x \in \phi^{-1}(p)} \operatorname{sgn}(J_\phi(x)). \tag{20}$$

(b) Let $\phi \in C^1(\bar{D})$ and $p \notin \phi(\partial D)$ such that $p \in \phi(Z_\phi)$. We define the degree of ϕ at p with respect to D , to be the number $\deg(\phi, D, q)$ for any $q \notin \phi(Z_\phi) \cup \phi(\partial D)$ such that $|p - q| < \operatorname{dist}(p, \phi(\partial D))$.

(c) Let $\phi \in C(\bar{D})$ and $p \in R^n \backslash \phi(\partial D)$. We define $\deg(\phi, D, p)$, the degree of ϕ at p with respect to D, to be $\deg(\psi, D, p)$ for any $\psi \in C^1(\bar{D})$ such that $|\psi(x) - \phi(x)| < \operatorname{dist}(p, \phi(\partial D))$ for all $x \in \bar{D}$.

The following list summarizes some properties most frequently used.

Theorem 3.2 *Suppose that $\phi \in C(\bar{D})$ and $p \notin \phi(\partial D)$. Then the following properties hold:*

(a) (Normalization) If $p \in D$ then $\deg(I, D, p) = 1$, where I is the identity mapping.
(b) (Existence) If $\deg(\phi, D, p) \neq 0$ then there is $x \in D$ such that $\phi(x) = p$.
(c) (Additivity) Suppose that D_1 and D_2 are disjoint open sets of D. If $p \notin \phi(\bar{D} \backslash (D_1 \cup D_2)$ then

$$\deg(D, f, p) = \deg(\phi, D_1, p) + \deg(\phi, D_2, p).$$

(d) (Homotopy invariance) Suppose that $H : [0, 1] \times D \to R^n$ is continuous. If $p \notin H(t, \partial D)$ for all $t \in [0, 1]$ then $\deg(H(t, .), D, p)$ is independent of t.
(e) (Excision) If D_0 is a closed set of D and $p \notin \phi(D_0)$ then $\deg(\phi, D, p) = \deg(\phi, D \backslash D_0, p)$.

Let us recall that x_0 is an isolated solution of an equation $\phi(x) = 0$ if there exists a bounded open neighborhood G of x_0 such that x_0 is the unique solution in $\bar{G}$. In particular, we have $0 \notin \phi(\partial G)$. Assume that G_1, G_2 are open neighborhoods of x_0 in G. By excision, we have $\mathrm{d}(\phi, G_1, 0) = \mathrm{d}(\phi, G_2, 0)$. The common value $\mathrm{d}(\phi, Q, 0)$ for open neighborhoods $Q \subset G$ of x_0 is called the index of ϕ at the isolated solution x_0 and denoted by $\operatorname{Ind}(\phi, x_0)$.

We now return to problem Eq. (1). For each $\xi \in C_+^*$ and $\rho > 0$ we define a mapping $F_{\xi,\rho}$ by the formula

$$F_{\xi,\rho}(\mu, \lambda, x) = x - \Pi_{K(\lambda)}\left(x - \rho \sum_{i=1}^m \xi_i f_i(\mu, x)\right),$$

where $\Pi_{K(\lambda)}(z)$ is the metric projection of a point $z \in R^n$ onto the set $K(\lambda)$. It is well known that $x \in S_\xi(\mu, \lambda)$ if and only if $0 = F_{\xi,\rho}(\mu, \lambda, x)$ for some $\rho > 0$. Hence from Lemma 2.1 we see that $x_0 \in S(\mu_0, \lambda_0)$ if and only if there exists $\xi^0 \in C_+^*$ such that $F_{\xi^0,\rho}(\mu_0, \lambda_0, x_0) = 0$. We shall call ξ^0 a functional corresponding to x_0.

We have the following result.

Theorem 3.3 *Suppose $x_0 \in S(\mu_0, \lambda_0)$ is an isolated solution, ξ^0 is a functional corresponding to x_0, X_0 is a bounded open neighborhood of x_0 and $M_0 \times \Lambda_0$ is a neighborhood of (μ_0, λ_0). Let $f_i : M_0 \times X_0 \to R^n$ be continuous mappings and $K : \Lambda_0 \to 2^{R^n}$ be a multifunction which satisfy conditions:*

(i) the map $\pi : \Lambda_0 \times X_0 \to R^n$ defined by $\pi(\lambda, z) = \Pi_{K(\lambda)}(z)$ is continuous;

(ii) *there exists* $\rho_0 > 0$ *such that*

$$\mathrm{Ind}(F_{\xi_0,\rho}(\mu_0, \lambda_0, .), x_0) \neq 0 \; \forall \rho \in (0, \rho_0].$$

Then there exist a neighborhood U_0 of μ_0, a neighborhood V_0 of λ_0 and an open bounded neighborhood Q_0 of x_0 such that the following assertions are fulfilled:

(a) *The solution map $S : U_0 \times V_0 \to 2^{Q_0}$ has nonempty values.*
(b) *The solution map S is lower semicontinuous at (μ_0, λ_0).*

Proof Choose $\rho \in (0, \rho_0]$ sufficiently small such that

$$x_0 - \rho \sum_{i=1}^{m} \xi_i^0 f_i(\mu_0, x_0) \in X_0.$$

By the continuity of f_i, there exists a neighborhood $M_1 \subset M_0$ of μ_0 and a neighborhood $X_1 \subset X_0$ of x_0 such that

$$x - \rho \sum_{i=1}^{m} \xi_i^0 f_i(\mu, x) \in X_0$$

for all $x \in X_1$ and $\mu \in M_1$. By (i), the map $F_{\xi^0,\rho}(\mu, \lambda, x)$ is continuous on $M_1 \times \Lambda_0 \times X_1$. Recall that

$$F_{\xi^0,\rho}(\mu, \lambda, x) = x - \Pi_{K(\lambda)}\left(x - \rho \sum_{i=1}^{m} \xi_i^0 f_i(\mu, x)\right).$$

From (ii) there exists a bounded open neighborhood $X_2 \subset X_1$ of x_0 such that the equation $F_{\xi^0,\rho}(\mu_0, \lambda_0, x) = 0$ has the unique solution x_0 in $\bar{X}_2$. Besides,

$$\mathrm{d}(F_{\xi^0,\rho}(\mu_0, \lambda_0, \cdot), X_2, 0) \neq 0.$$

This implies that for each $w \in \partial X_2$, $F_{\xi^0,\rho}(\mu_0, \lambda_0, w) \neq 0$. Putting $u_w = F_{\xi^0,\rho}(\mu_0, \lambda_0, w)$ and $r_w = \frac{1}{2}\|u_w\| > 0$, we see that $0 \notin B(u_w, r_w)$. By the continuity of $F_{\xi^0,\rho}(\mu_0, \lambda_0, w)$, there exist a neighborhood X_w of w and a neighborhood $U_w \times V_w$ of (μ_0, λ_0) such that $F_{\xi^0,\rho}(\mu, \lambda, x) \in B(u_w, r_w)$ for all $(\mu, \lambda) \in U_w \times V_w$ and $x \in X_w$. Since ∂X_2 is compact, there exist points $w_1, w_2, ..., w_k$ such that $\partial X_2 \subseteq \cup_{i=1}^{k} X_{w_i}$. Put $Q_0 = X_2$, $U_0 = \cap U_{w_i}$ and $V_0 = \cap V_{w_i}$. We want to show that Q_0, U_0 and V_0 satisfy the conclusion of the theorem.

Fixing any $(\mu, \lambda) \in U_0 \times V_0$, we consider the homotopy $H : [0, 1] \times \bar{X}_2 \to 2^{R^n}$ defined by $H(t, x) = (1 - t)F_{\xi^0,\rho}(\mu_0, \lambda_0, x) + tF_{\xi^0,\rho}(\mu, \lambda, x)$. For each $w \in \partial X_2$ then $w \in X_{w_i}$ for some i. Since $(\mu, \lambda) \in U_{w_i} \times V_{w_i}$, we have $F_{\xi^0,\rho}(\mu, \lambda, w) \in B(u_{w_i}, r_{w_i})$ and $F_{\xi^0,\rho}(\mu_0, \lambda_0, w) \in B(u_{w_i}, r_{w_i})$. By the convexity of $B(u_{w_i}, r_{w_i})$ we have

$$(1 - t)F_{\xi^0,\rho}(\mu_0, \lambda_0, x) + tF_{\xi^0,\rho}(\mu, \lambda, x) \in B(u_{w_i}, r_{w_i}).$$

As $0 \notin B(u_{w_i}, r_{w_i})$, it follows that

$$0 \notin H(t, w) = (1 - t)F_{\xi^0,\rho}(\mu_0, \lambda_0, x) + tF_{\xi^0,\rho}(\mu, \lambda, x)$$

for all $t \in [0, 1]$ and $w \in \partial X_2$. By (d) of Theorem 3.2, we have

$$\mathrm{d}(F_{\xi^0,\rho}(\mu, \lambda, \cdot), X_2, 0) = \mathrm{d}(F_{\xi^0,\rho}(\mu_0, \lambda_0, \cdot), X_2, 0) \neq 0.$$

By (b) in Theorem 3.2, we can find a point $x = x(\mu, \lambda) \in X_2$ such that $F_{\xi^0,\rho}(\mu, \lambda, x) = 0$. This implies that $x(\mu, \lambda) \in S_{\xi^0}(\mu, \lambda) \cap X_2 \subset S(\mu, \lambda) \cap Q_0$. Hence assertion (a) is proved.

Let us prove (b). Suppose G is an open set in Q_0 such that $S(\mu_0, \lambda_0) \cap G \neq \emptyset$. Note that $G = Q_0 \cap D$, where D is an open set in R^n. By the uniqueness of x_0 in $Q_0(= X_2)$, we have $x_0 \in G$. By using the same procedure as in the proof of assertion (a), we can show that there exists a neighborhood $\hat{X}_2 \subset G$ and a neighborhood $\hat{U} \times \hat{V} \subset U_0 \times V_0$ of (μ_0, λ_0) such that $S_{\xi^0}(\mu, \lambda) \cap \hat{X}_2 \neq \emptyset$ for all $(\mu, \lambda) \in \hat{U} \times \hat{V}$. Note that $\cup_{\xi \in C_+^*} S_\xi(\mu, \lambda) = S(\mu, \lambda)$. Hence we have $S(\mu, \lambda) \cap G \neq \emptyset$ for all $(\mu, \lambda) \in \hat{U} \times \hat{V}$. Consequently, S is lower semicontinuous at (μ_0, λ_0). The proof of the theorem is complete. □

In the above theorem, condition (i) is a key hypothesis. In order to apply it one needs to verify this requirement. The following proposition provides a sufficient condition for the fulfillment of condition (i).

Proposition 3.4 *Suppose K has the Aubin property of order $\alpha > 0$ at (λ_0, x_0). Then there exist neighborhoods X_0 of x_0 and V_0 of λ_0 such that the mapping $\pi : V_0 \times X_0 \to R^n$ defined by $\pi(\lambda, z) = \Pi_{K(\lambda)}(z)$ is continuous on $V_0 \times X_0$.*

Proof According to [9, Theorem 3.1] (see also [17, Lemma 1.1]), there exists a neighborhood X_0 of x_0 and V_0 of λ_0 and a constant $k_0 > 0$ such that

$$\|\Pi_{K(\lambda)}(z) - \Pi_{K(\lambda')}(z)\| \leq kd(\lambda, \lambda')^{\frac{\alpha}{2}} \tag{21}$$

for all $z \in X_0$ and $\lambda, \lambda' \in V_0$. It remains to prove that the map $\pi(\lambda, z)$ is continuous on $V_0 \times X_0$. In fact, taking any $(\lambda, z) \in V_0 \times X_0$ and assume that $z_n \to z, \lambda_n \to \lambda$. We want to show that $\pi(\lambda_n, z_n) \to \pi(\lambda, z)$. From Eq. (21) we have the following estimation

$$\begin{aligned}
\|\pi(\lambda_n, z_n) - \pi(\lambda, z)\| &= \|\Pi_{K(\lambda_n)}(z_n) - \Pi_{K(\lambda)}(z)\| \\
&\leq \|\Pi_{K(\lambda_n)}(z_n) - \Pi_{K(\lambda)}(z_n)\| + \|\Pi_{K(\lambda)}(z_n) - \Pi_{K(\lambda)}(z)\| \\
&\leq kd(\lambda_n, \lambda')^{\frac{\alpha}{2}} + \|z_n - z\|.
\end{aligned}$$

Hence $\|\pi(\lambda_n, z_n) - \pi(\lambda, z)\| \to 0$ as $n \to \infty$. Consequently, π is continuous at (λ, z). Since (λ, z) is arbitrary, π is continuous on $V_0 \times X_0$. □

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